Asymptotic Cramér–Rao bounds for two closely spaced damped cisoids

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Abstract

Simple closed-form expressions are presented for the Cramér–Rao bounds (CRBs) for estimating the parameters of two closely spaced damped cisoids in complex additive white Gaussian noise in the case of low damping and a sufficiently large number of data samples. The expressions provide useful insight into the behaviour of the CRBs with the frequency separation and the damping factors of the cisoids.

1. Introduction

The Cramér–Rao bound (CRB) specifies a lower bound on the variance of any unbiased estimator. Accordingly, the CRB frequently is used as a yardstick for assessing performance of practical estimators. The CRBs for estimating the parameters (amplitudes, phases, damping factors, and frequencies) of superimposed damped complex sinusoids (cisoids) in complex additive white Gaussian noise (AWGN) are derived, for example, in [1,2]. The dependence of the CRBs on the signal parameters, such as, the frequency separations between the cisoids, the damping factors of the cisoids, and the number of data samples is of interest. Due to the complexity of the CRB expressions, which involve matrix inversions, these dependences of the CRBs have been explored numerically rather than analytically, even for the simplest case of a single damped cisoid in complex AWGN. Indeed, few analytical results on the relationships between the CRBs and the signal parameters are available. It is known, for example, that the CRBs are independent of the phases of the cisoids and depend on the frequencies of the cisoids only through their differences. Also, the CRBs for a particular cisoid are inversely proportional to the signal-to-noise ratio for that cisoid except for the amplitude CRB that is proportional to the variance of the noise [1,2].

Recently, Swingler demonstrated that the CRBs for the case of one damped cisoid in complex AWGN could be approximated by simple expressions that provide much insight into the behaviour of the CRBs against the signal parameters [3]. In this note, we consider the more complicated case
of two damped cisoids in complex AWGN. This two-signal data model is a prototype of multiple signal time-series data models and extensively is used for testing the performance of the so-called high-resolution spectrum estimation algorithms where the frequency separation between the two cisoids is assumed to be less than the resolution limit of the periodogram (the Fourier limit). We present simple closed-form expressions for the asymptotic (as the frequency separation goes to zero) CRBs under the assumptions of low damping and a sufficiently large number of data samples. The expressions give the dependence of the CRBs on the frequency separation and the damping factors in a simple way. Moreover, it turns out that the asymptotic expressions closely approximate the exact CRBs for the whole range of the frequency separation under the abovementioned assumptions.

2. Model description and assumptions

We consider the data model

\[ y(t) = \sum_{i=1}^{2} z_i e^{-\beta_i t} e^{j(\omega_i t + \phi_i)} + e(t), \quad t = 1, \ldots, N, \quad (1) \]

where \( z_i \) is the amplitude, \( \beta_i \) is the damping factor, \( \omega_i \in (0, 2\pi) \) is the frequency, \( \phi_i \) is the phase of the \( i \)th cisoid, \( i = 1, 2 \), \( e(t) \) represents the zero-mean complex AWGN with variance \( \sigma^2 \), and \( N \) is the total number of data samples.

We make the following assumptions on the number of data samples \( N \) and on the damping factors of the two cisoids:

(A1) \( N \) is such that \( N e^{-\beta_i N} \ll 1 \) for \( i = 1, 2 \),

(A2) \( \beta_i \ll 1 \) for \( i = 1, 2 \).

Assumption A1 means that the signal is measured until it has faded away, that is, the number of data samples should be sufficiently large. Assumption A2 means that the damping of each cisoid is low.

3. Asymptotic bound expressions

Let CRB\((z_i)\), CRB\((\phi_i)\), CRB\((\beta_i)\) and CRB\((\omega_i)\) denote the CRB for, respectively, the amplitude, phase, damping factor and frequency of the \( i \)th cisoid, \( i = 1, 2 \). Then, the following relations exist among these CRBs [1]:

\[
\text{CRB}(\phi_i) = \frac{\text{CRB}(z_i)}{\sigma^2},
\]

\[
\text{CRB}(\omega_i) = \text{CRB}(\beta_i).
\]

Thus, we will present the asymptotic bound expressions for only amplitude and frequency estimation.

It is shown in Appendix A that the asymptotic amplitude and frequency CRBs for the model in (1) as \( \delta \omega \to 0 \) where \( \delta \omega \) denotes the frequency separation between the two cisoids under the assumptions A1 and A2 can be expressed as follows:

\[
\text{CRB}(z_i) = \frac{2\sigma^2 \beta_i (\delta \omega^2 + (\beta_i + \beta_j)^2)}{(\delta \omega^2 + (\beta_i - \beta_j)^2)^3} \\
\times ((\delta \omega^2 + (\beta_i + \beta_j)^2)^2 \\
+ 4\beta_i \beta_j (\delta \omega^2 + (\beta_i + \beta_j)^2) \\
- 16\beta_i^2 \beta_j (\beta_i - \beta_j)) \quad (2)
\]

for amplitude estimation, and

\[
\text{CRB}(\omega_i) = \frac{4\beta_i^3 (\delta \omega^2 + (\beta_i + \beta_j)^2)^2}{\text{SNR}_i (\delta \omega^2 + (\beta_i - \beta_j)^2)^2} \quad (3)
\]

for frequency estimation. Here \( i \neq j = 1, 2 \) and \( \text{SNR}_i \) denotes the signal-to-noise ratio for the \( i \)th cisoid, defined as \( \text{SNR}_i = \sigma^2 / \sigma^2 \).

Clearly, the asymptotic results in (2) and (3) give the dependence of the CRBs on the frequency separation \( \delta \omega \) and the damping factors \( \beta_1 \) and \( \beta_2 \) in a simple way. Note that the asymptotic bounds are independent of the number of data samples \( N \). This is due to the assumption A1 that the signal is measured until it has faded away.

It can be seen from the results in (2) and (3) that the behaviour of the CRBs for small frequency separations \( \delta \omega \) for the case of equal damping factors differs from that for the case of unequal damping factors. Eq. (2) shows that the amplitude (phase) CRB for the \( i \)th cisoid \((i = 1, 2)\) is proportional to \( \beta_i (\delta \omega / \beta_i)^{-6} \) for \( \beta_1 = \beta_2 \) while it is proportional to \( \beta_i (\beta_i - \beta_2)^{-6} \) (which is independent of \( \delta \omega \) for \( \beta_1 \neq \beta_2 \) as \( \delta \omega \to 0 \) under A1 and A2).

Eq. (3) shows that the frequency (damping factor) CRB for the \( i \)th cisoid \((i = 1, 2)\) is proportional to \( \beta_i^3 (\delta \omega / \beta_i)^{-4} \) for \( \beta_1 = \beta_2 \) while it is proportional to \( \beta_i^3 (\beta_i - \beta_2)^{-4} \) for \( \beta_1 \neq \beta_2 \) as \( \delta \omega \to 0 \) under A1 and A2.

Note that for the case of equal damping factors, the inverse power dependence of the asymptotic bounds on \( \delta \omega \) is through the ratio \( \delta \omega / \beta_i \) indicating
that the bounds will be large provided that $\delta \omega / \beta_i$, rather than $\delta \omega$, is small. This shows the importance of having a large value for the quantity $1/\beta_i$ when $\delta \omega$ is small in this case.

In [4], Wigren and Nehorai have derived simple expressions for signal parameter CRBs for the more general data model consisting of multiple damped sinusoids in AWGN for both real and complex valued data cases under the same assumptions of low damping and a large number of data samples, and an additional assumption which implies, for our model, that the frequency separation $\delta \omega$ between the two cisoids is "large." Specifically, they assume

$$\delta \omega \gg \beta_1 + \beta_2.$$  \hspace{1cm} (4)

With this additional assumption the CRBs for the $i$th cisoid ($i = 1, 2$) are given by [4]

$$\text{CRB}(\alpha_i) = 2\sigma^2 \beta_i,$$  \hspace{1cm} (5)

$$\text{CRB}(\omega_i) = \frac{4\beta_i^3}{\text{SNR}_i}.$$  \hspace{1cm} (6)

Note that these large-frequency-separation results are independent of the frequency separation $\delta \omega$.

Although our asymptotic results in (2) and (3) are derived for small frequency separations, use of the condition (4) in (2) and (3) gives that they are also valid for large frequency separations. Indeed, examples indicate that the simple expressions in (2) and (3) closely approximate the exact CRBs for

Fig. 1. Normalised exact and asymptotic CRB for amplitude estimation of two cisoids having frequency separation $\delta \omega$, damping factors $\beta_1$ and $\beta_2$, observed by $N = 100$ uniformly spaced samples: (a) $\beta_1 = 0.1, \beta_2 = 0.1$; and (b) $\beta_1 = 0.1, \beta_2 = 0.15$.

Fig. 2. Normalised exact and asymptotic CRB for frequency estimation of two cisoids having frequency separation $\delta \omega$, damping factors $\beta_1$ and $\beta_2$, observed by $N = 100$ uniformly spaced samples: (a) $\beta_1 = 0.1, \beta_2 = 0.1$; and (b) $\beta_1 = 0.1, \beta_2 = 0.15$. 

the whole range of the frequency separation under the assumptions A1 and A2.

Example. Consider the case of \( N = 100 \) data samples with two different choices of the damping factors:

(a) \( \beta_1 = 0.1, \beta_2 = 0.1, \) and
(b) \( \beta_1 = 0.1, \beta_2 = 0.15, \)

representing the cases of equal and unequal damping factors, respectively. Fig. 1 shows the amplitude CRB representing the cases of equal and unequal damping factors.

\[ \text{Fig. 1 shows the amplitude CRB} \]

…

The asymptotic bounds in [4], repeated here in (5) and (6), are also shown in the figures by the vertical dotted lines for the purpose of comparison. The difference between the bounds is shown in the figures and these horizontal lines describes the impact of the presence of the second cisoid on the CRB of the “well-separated” version of the first.

4. Conclusions

We have derived simple expressions for the asymptotic CRBs for the problem of estimating the parameters of two closely spaced damped cisoids in complex additive white Gaussian noise under the assumptions of low damping and a sufficiently large number of data samples. The expressions closely approximate the results for the whole range of the frequency separation between the cisoids. The results are important in that they provide insight into the behaviour of the CRBs against the frequency separation and the damping factors of the cisoids.

Appendix A

In this appendix, we derive the asymptotic CRB expressions in (2) and (3). Deducing the corresponding exact CRBs from the results given in [1] or [2], upon which performing some straightforward albeit tedious calculations, the exact CRBs can be expressed in closed (nonmatrix) form as follows:

\[ \text{CRB}(\omega_i) = \frac{1}{SNR_i \cdot N^3} \frac{a_i}{2(a_i b_i - b_i^2 \pm \delta \omega)}, \quad (A.1) \]

\[ \text{CRB}(\omega_o) = \frac{1}{SNR_i \cdot N^3} \frac{a_i}{2(a_i b_i - b_i^2 \pm \delta \omega)}, \quad (A.2) \]

where

\[ a_i = \Gamma_{i,0} - \frac{(C_1 + S_1 \delta \omega_i - 2(C_0 S_1 + S_0 \delta \omega_i) \Gamma_{j,1} + (C_0 S_1 + S_0 \delta \omega_i) \Gamma_{j,2}}{\Gamma_{j,0} \Gamma_{j,2} - \Gamma_{j,1}^2} \]

(A.3)

Here \( i \neq j = 1, 2, \) and for \( r = 0, 1, 2 \)

\[ \Gamma_{i,r} = \frac{1}{N^{r+1}} \sum_{i=1}^{N} t^r e^{-2\beta_i t}, \quad (A.7) \]

\[ C_r = \frac{1}{N^{r+1}} \sum_{i=1}^{N} t^r e^{-(\beta_1 + \beta_2) t} \cos(\delta \omega t), \quad (A.8) \]

\[ S_r = \frac{1}{N^{r+1}} \sum_{i=1}^{N} t^r e^{-(\beta_1 + \beta_2) t} \sin(\delta \omega t), \quad (A.9) \]
Under assumption A1, the CRB expressions given by (A.1)–(A.9) simplify to

\[
\sigma^2(z_i^2 - 1)(1 + z_i^2 z_j^2 - 2z_i z_j \cos(\delta \omega)) \\
\times \{4 - 3z_i^2 + z_i^4 - 7z_j^2 + 4z_j^4 + 19z_i^2 z_j^2 + 9z_i^4 z_j^4 - 5z_i^4 z_j^4 - z_i^6 z_j^2 - 11z_i^2 z_j^4 \}
\]

\[
\text{CRB}(z_i) = \frac{1}{2(z_i^2 + z_j^2 - 2z_i z_j \cos(\delta \omega))^2},
\]

\[
\text{CRB}(\omega_i) = \frac{1}{\text{SNR}} \frac{z_i^2(z_i^2 - 1)^3(1 + z_i^2 z_j^2 - 2z_i z_j \cos(\delta \omega))^2}{2(z_i^2 + z_j^2 - 2z_i z_j \cos(\delta \omega))^2},
\]

where we have utilised the shorthand \(z_i = \exp(\beta_i)\), \(i = 1, 2\).

Expanding the numerator and the denominator of (A.10) and (A.11) into the Taylor series about \(\delta \omega = 0\) shows that for small \(\delta \omega\)

\[
\text{CRB}(z_i) = \frac{\sigma^2(p_{z_i,0} + p_{z_i,2} \delta \omega^2 + p_{z_i,4} \delta \omega^4 + p_{z_i,6} \delta \omega^6 + O(\delta \omega^8))}{q_{z_i,0} + q_{z_i,2} \delta \omega^2 + q_{z_i,4} \delta \omega^4 + q_{z_i,6} \delta \omega^6 + O(\delta \omega^8)},
\]

\[
(A.12)
\]

\[
\text{CRB}(\omega_i) = \frac{p_{\omega_i,0} + p_{\omega_i,2} \delta \omega^2 + p_{\omega_i,4} \delta \omega^4 + p_{\omega_i,6} \delta \omega^6 + O(\delta \omega^8)}{q_{\omega_i,0} + q_{\omega_i,2} \delta \omega^2 + q_{\omega_i,4} \delta \omega^4 + q_{\omega_i,6} \delta \omega^6 + O(\delta \omega^8)},
\]

\[
(A.13)
\]

where the coefficients \(p_{z_i,m} \), \(q_{z_i,m} \), \(m = 0, 2, 4, 6\), and \(p_{\omega_i,m} \), \(q_{\omega_i,m} \), \(n = 0, 2, 4\) depend \(n = 0, 2, 4\) depend on \(\beta_1\) and \(\beta_2\). For the sake of brevity, the coefficients are not provided here. Nevertheless, for verification reasons, we would like to present only the first coefficients \(p_{z_i,0} \), \(q_{z_i,0} \), \(p_{\omega_i,0} \), and \(q_{\omega_i,0} \) as an example, which are

\[
p_{z_i,0} = 8(2\beta_i - \beta_j)^2(\beta_i + \beta_j)^4 + 4\beta_i \beta_j (\beta_i, \beta_j)^2
\]

\[
(A.14)
\]

\[
q_{z_i,0} = 2(\beta_i - \beta_j)^6,
\]

\[
(A.15)
\]

We then use the assumption A2 by expressing each coefficient in terms of Taylor series about \(\beta_1 = 0\) and \(\beta_2 = 0\), and retaining adequate number of terms. Namely, we keep the terms with the order less than or equal to \(O(\beta_i^{1+k} \beta_j^l)\) and \(O(\beta_i^k \beta_j^l)\) for the coefficients \(p_{z_i,m} \) and \(q_{z_i,m} \), respectively, where \(i \neq j = 1, 2\) and \(k + l = 6 - m\) and with the order less than or equal to \(O(\beta_i^{1+k} \beta_j^l)\) and \(O(\beta_i^k \beta_j^l)\) for the coefficients \(p_{\omega_i,n} \) and \(q_{\omega_i,n} \), respectively, for which \(i \neq j = 1, 2\) and \(k + l = 4 - n\) hold. By this way, the first coefficients in (A.14)–(A.17), for example, take the form

\[
p_{z_i,0} = 4\beta_i (\beta_i^6 - 6\beta_i^5 \beta_j + 15\beta_i^4 \beta_j^2 + 60\beta_i^3 \beta_j^3
\]

\[
+ 47\beta_i^2 \beta_j^4 + 10\beta_i \beta_j^5 + \beta_j^6)
\]

\[
= 4\beta_i (\beta_i + \beta_j)^2 (\beta_i + \beta_j)^4 + 4\beta_i \beta_j (\beta_i + \beta_j)^2
\]

\[
- 16\beta_i^2 \beta_j (\beta_i - \beta_j)),
\]

\[
(A.18)
\]

\[
q_{z_i,0} = 2(\beta_i - \beta_j)^6,
\]

\[
(A.19)
\]

\[
p_{\omega_i,0} = 8\beta_i^2 (\beta_i + \beta_j)^4,
\]

\[
(A.20)
\]

\[
q_{\omega_i,0} = 2(\beta_i - \beta_j)^4.
\]

Obtaining the approximate expressions for the coefficients in this way, and substituting them into (A.12) and (A.13) gives (2) and (3).

References


