Nonmatrix Cramér–Rao Bound Expressions for High-Resolution Frequency Estimators

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Abstract—Analytical expressions are derived for the Cramér–Rao (CR) lower bound on the variance of frequency estimates for the two-signal time-series data models consisting of either one real sinusoid or two complex sinusoids in white Gaussian noise. The expressions give the bound in terms of the signal-to-noise ratio (SNR), the number N of data samples, and a function dependent on the frequency separation and the initial phase difference between the two signal components of each model. The bounds are examined as the phase difference is varied, and the largest and smallest bound expressions and the corresponding critical values of the phase difference are obtained.

The exact expressions are analyzed for the case of small frequency separations δω. It is found that the largest bound is proportional to (N · δω)−1/N and that the smallest bound is proportional to (N · δω)−2/N · SNR for small δω. Examples indicate that the small δω results closely approximate the exact ones whenever the frequency separation is smaller than the Fourier resolution limit. Based on the asymptotic results, it is found that the threshold SNR at which an unbiased estimator can resolve the two signal frequencies is at least proportional to (N · δω)−1/N for the worst phase difference case and to (N · δω)−1/N for the best phase difference case for small δω.

The results are applicable to the general case of sampling where the samples are taken at arbitrary instants.

I. INTRODUCTION

The two-signal time-series data models consisting of either a single real sinusoid in real white Gaussian noise (the real model) or two complex sinusoids in complex white Gaussian noise (the complex model) extensively are used for testing the performance of high-resolution frequency estimation algorithms where the separation of the two signal frequencies present in each model is assumed to be less than the resolution limit of the periodogram (the Fourier limit).

It is known that under weak conditions, the variance of any unbiased estimate is always bounded below by the Cramér–Rao (CR) bound [1], [2]. Its study, therefore, provides the ultimate performance limits for the data models, independent of the estimation algorithm used. For each model, the dependence of the bound on the frequency separation, the initial phase difference between the two signal components, the signal-to-noise ratio (SNR), and the number of data samples is of interest.

Evaluation of the CR bound requires inverting the applicable Fisher information matrix J. Due to the complexity of J in the multiple signal case, the CR bounds typically are computed numerically rather than analytically [3], [4]. As a result, the aforementioned dependences of the bounds have been explored via simulation, even for the two-signal cases.

Recently, Stoica and Nehorai derived a useful expression for calculating the CR frequency bounds for multiple complex sinusoids in complex white Gaussian noise [5]. Lee analyzed the Stoica and Nehorai formula for the case of small frequency separations of the signals and provided simple expressions for the asymptotic bounds [6]. The expressions apparently lead to very high SNR values for the resolvability of the signals when their separation is small, thus rendering high-resolution frequency estimation algorithms impractical. However, it is reported in [7] that the expressions may not apply to the common case of single experiment data where only one sample function of the time series is utilized and that good resolution properties can be expected at reasonable SNR values in this case.

In this paper, exact analytical expressions for the CR frequency bound are derived in nonmatrix forms for both real and complex two-signal data models in the single-experiment case. For each model, the bound is given in terms of the SNR, the number of data samples, and a function dependent on the frequency separation and the initial phase difference between the two signal components. A special decomposition of the Fisher information matrix for each model results in the bound expression, which gives the dependence of the bound on the initial phase difference in a simple way. As a result, explicit expressions for the largest and smallest frequency bounds and for the corresponding critical values of the phase difference are obtained for both models.

The exact expressions are analyzed for the case of small frequency separations, and simple expressions for the largest and smallest bounds and for the critical phases are developed. It is found that the largest bound is given by

\[ \text{var} \{ \hat{\omega}_i \} \geq \frac{M}{\text{SNR}_i \cdot N^3 (N \cdot \delta \omega)^4} + O \{ (N \cdot \delta \omega)^{-2} \} \]  

and that the smallest bound is given by

\[ \text{var} \{ \hat{\omega}_i \} \geq \frac{m}{\text{SNR}_f \cdot N^3 (N \cdot \delta \omega)^2} + O \{ 1 \} \]  

for small \( \delta \omega \), where

- \( \text{SNR}_f \) SNR for the \( i \)th signal \( i = 1, 2 \);
- \( \delta \omega \) frequency difference between the two signals;
- \( N \) number of data samples;
- \( M, m \) suitable constants for each model.
The results in (1) and (2) show that the inverse power dependence of the bound on the frequency separation $\delta \omega$ occurs through the product $(N \cdot \delta \omega)^{-6}/N$ for small $\delta \omega$, implying the importance of a sufficient number of samples in this case.

Based on the results in (1) and (2), it is found that the threshold SNR at which an unbiased estimator can resolve the two signals with high probability is at least proportional to $(N \cdot \delta \omega)^{-4}/N$ for the worst-phase case and to $(N \cdot \delta \omega)^{-1}/N$ for the best-phase case when the frequency separation is small. The fact that the bound depends on the inverse powers of $(N \cdot \delta \omega)$ for small $\delta \omega$ in the single experiment case considered herein results in rather low threshold SNR values when the number of data samples is sufficiently large.

The results are presented for the common case of uniform sampling and applicable to the general case of sampling where the samples are taken at arbitrary instants.

The paper is organized as follows. Sections II–V consider the real model. Section II describes the model. Section III derives the nonmatrix frequency bound formula and examines the initial phase dependence of the bound. Section IV gives the asymptotic (as the frequency separation goes to zero) results. Section V addresses the problem of resolving the two signals for small separations. Section VI develops the results for the complex model.

II. REAL DATA MODEL

We consider first the real time-series data that consist of a single real sinusoid and real zero-mean white Gaussian noise $\sigma(t)$ with variance $\sigma^2$

$$y(t) = \alpha_0 \cos(\omega_0 t + \varphi_0) + \epsilon(t) \quad t = n \cdots n + N - 1 \quad (3)$$

where the positive amplitude $\alpha_0$, the initial phase angle $\varphi_0$, the normalized frequency $\omega_0$ of the sinusoid, and the variance $\sigma^2$ of the noise are assumed unknown parameters to be estimated from the $N$ data samples.

In (3), the common case of uniform sampling is assumed where $n$ denotes the first value of the sample index $t$. If the number of samples $N$ is odd and $n = -(N - 1)/2$, then the sampling is symmetric. It turns out that the symmetric sampling case greatly simplifies the results.

III. CR FREQUENCY BOUND

The CR theorem gives a lower bound on the variance of any unbiased estimator $\hat{\omega}_0$ of the frequency $\omega_0$ of the sinusoid

$$\text{var} \{\hat{\omega}_0\} = E\{(\hat{\omega}_0 - \omega_0)^2\} \geq B_0 \quad (4)$$

where $B_0$ is the diagonal term corresponding to the parameter $\omega_0$ of the inverse Fisher information matrix $J^{-1}$. We prove the following theorem, which gives a nonmatrix expression for the CR frequency bound $B_0$.

**Theorem 1:** Let the data be given by (3). Then, the CR frequency bound $B_0$ is given by

$$B_0 = \frac{1}{\text{SNR} \cdot N^3 K_0 + K_C \cdot \cos(2\varphi_0) + K_S \cdot \sin(2\varphi_0)} \quad (5)$$

SNR denotes the signal-to-noise ratio for the sinusoid

$$\text{SNR} = \frac{\alpha_0^2}{2\sigma^2} \quad (6)$$

$N$ denotes the number of data samples, and $K_0, K_C, K_S$ are given by (7)–(9), shown at the bottom of the page, where

$$\Gamma_r = \frac{1}{N^{r+1}} \sum_{t=1}^{n+N-1} t^r \quad (10)$$

$$C_r = \frac{1}{N^{r+1}} \sum_{t=1}^{n+N-1} t^r \cos(\delta \omega \cdot t) \quad (11)$$

$$S_r = \frac{1}{N^{r+1}} \sum_{t=1}^{n+N-1} t^r \sin(\delta \omega \cdot t) \quad (12)$$

$r = 0, 1, 2$, and $\delta \omega$ denotes the separation of the two signal frequencies present in the real model

$$\delta \omega = 2 \cdot \omega_0 \quad (13)$$

**Proof:** The proof is given in Appendix A.

The first term on the right-hand-side of (5) gives the dependence of the CR frequency bound on the SNR and the number of data samples $N$. The bound is inversely proportional to the SNR and decreases as the cube power of $N$. The second term of (5) gives the dependence of the bound on the frequency separation and the initial phase.

Note that although the uniform sampling is assumed, (5) becomes valid for any sampling instants when the summation indices of (10)–(12) are changed accordingly.

A. CR Bound Versus Phase

For fixed SNR, the result (5) gives the dependence of the CR frequency bound $B_0$ on the initial phase $\varphi_0$ of the sinusoid in a simple way. We now examine the CR bound (5) as $\varphi_0$ varies.

$$K_0 = \frac{\Gamma_2 \cdot (\Gamma^2_0 - C_0^2 - S_0^2) - \Gamma_0 \cdot (\Gamma^2_1 + C_1^2 + S_1^2) + 2 \cdot \Gamma_1 \cdot (C_0 C_1 + S_0 S_1)}{\Gamma_0^2 \cdot C_0^2 - S_0^2} \quad (7)$$

$$K_C = \frac{-C_2 \cdot (\Gamma^2_0 - C_0^2 - S_0^2) - C_0 \cdot (\Gamma^2_1 + C_1^2 - S_1^2) + 2 \cdot C_1 \cdot (\Gamma_0 \Gamma_1 - S_0 S_1)}{\Gamma_0^2 \cdot C_0^2 - S_0^2} \quad (8)$$

$$K_S = \frac{S_2 \cdot (\Gamma^2_0 - C_0^2 - S_0^2) + S_0 \cdot (\Gamma^2_1 - C_1^2 + S_1^2) - 2 \cdot S_1 \cdot (\Gamma_0 \Gamma_1 - C_0 C_1)}{\Gamma_0^2 \cdot C_0^2 - S_0^2} \quad (9)$$
It follows from (5) that the bound is periodic in \( \varphi_0 \) with a period of \( \pi \). Thus, it is sufficient to consider the bound in any interval of length \( \pi \). Let us choose the interval

\[
I \equiv \{ \varphi_0 : \varphi_0 \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \}.
\]

(14)

It is important to determine the values of the initial phase angle \( \varphi_0 \) at which the frequency bound \( B_0 \) attains its maximum and minimum values. This can be done by taking the derivative of (5) with respect to \( \varphi_0 \) and setting it to zero. The results are given by the following corollary:

**Corollary 1:** The CR frequency bound \( B_0 \) as a function of the initial phase angle \( \varphi_0 \) has one maximum point and one minimum point in the interval \( I \) given by, respectively

\[
(\varphi_0)_{\text{max}} = \begin{cases} 
\phi & \text{if } K_C \geq 0 \\
\phi & \text{if } K_C < 0 
\end{cases}
\]

and

\[
(\varphi_0)_{\text{min}} = \begin{cases} 
\phi & \text{if } K_C \geq 0 \\
\phi - \text{sgn} \left( \frac{K_S}{K_C} \right) \cdot \frac{\pi}{2} & \text{if } K_C < 0 
\end{cases}
\]

(15), (16)

where

\[
\phi = \frac{1}{2} \arctan \left( \frac{K_S}{K_C} \right)
\]

(17)

and

\[
\text{sgn}(x) = \begin{cases} 
1 & x > 0 \\
-1 & x \leq 0 
\end{cases}
\]

(19)

Furthermore, the maximum and minimum values of \( B_0 \) are given by, respectively

\[
(B_0)_{\text{max}} = \frac{1}{\text{SNR} \cdot N^3} \frac{1}{K_0 - \sqrt{K_C^2 + K_S^2}}
\]

(20)

\[
(B_0)_{\text{min}} = \frac{1}{\text{SNR} \cdot N^3} \frac{1}{K_0 + \sqrt{K_C^2 + K_S^2}}
\]

(21)

Thus, the values of \( \varphi_0 \) given by (15) and (16) give, respectively, the largest and smallest CR bound on the variance of any unbiased estimator \( \hat{\omega}_0 \) of the frequency \( \omega_0 \). The determinant of the inverse Fisher information matrix is also a criterion for the worst and best CR bounds [8]. From the expression for the inverse Fisher information matrix \( J^{-1} \) derived in Appendix A, we see that \( \det(J^{-1}) \) as a function of \( \varphi_0 \) also attains its largest and smallest values at the same \( \varphi_0 \) values of (15) and (16), respectively.

In Theorem 1 and Corollary 1, we tacitly assumed that the Fisher information matrix \( J \) for the problem was invertible (positive definite). It can be shown that for \( N = 1 \), the common denominator term in (7)–(9) is zero; for \( N = 2 \), \( K_0 \) and thus, \( K_C \) and \( K_S \) are zero; for \( N = 3 \), the denominator term in (20) is zero. Accordingly, the following conditions on the number of samples \( N \) are necessary for the invertibility of \( J \)

\[
N > 2, \quad \text{and} \quad N = 3 \Rightarrow \varphi_0 \neq (\varphi_0)_{\text{max}}.
\]

(22)

**B. Symmetric Sampling**

The case where \( N \) is an odd number and the first sample index \( n = -(N-1)/2 \) greatly simplifies the results found to this point. In this case, it follows from (10)–(12) that

\[
\Gamma_1 = 0; \quad C_1 = 0; \quad S_0 = 0; \quad S_2 = 0
\]

(23)

so that the CR frequency bound is given by (5) with

\[
K_0 = \Gamma_2 - \Gamma_0 \cdot \frac{S_0^2}{\Gamma_0^2 - C_0^2}
\]

(24)

\[
K_C = -C_2 + C_0 \cdot \frac{S_0^2}{\Gamma_0^2 - C_0^2}
\]

(25)

\[
K_S = 0,
\]

(26)

The worst and best phase expressions (15) and (16) become

\[
(\varphi_0)_{\text{max}} = \begin{cases} 
\pi & \text{if } K_C > 0 \\
0 & \text{if } K_C < 0
\end{cases}
\]

(27)

\[
(\varphi_0)_{\text{min}} = \begin{cases} 
0 & \text{if } K_C > 0 \\
\pi & \text{if } K_C < 0
\end{cases}
\]

(28)

The largest and smallest bound expressions (20) and (21) become

\[
(B_0)_{\text{max}} = \frac{1}{\text{SNR} \cdot N^3} \frac{1}{K_0 - |K_C|}
\]

(29)

\[
(B_0)_{\text{min}} = \frac{1}{\text{SNR} \cdot N^3} \frac{1}{K_0 + |K_C|}
\]

(30)

**IV. Asymptotic CR Bound**

We next examine the CR bound (5) when the frequency separation \( \delta \omega \) is small. We do this by expressing the frequency-separation-dependent functions \( K_0, K_C, \) and \( K_S \) of (5) in terms of Taylor series about \( \delta \omega = 0 \) and identifying the dominant term of (5) as \( \delta \omega \to 0 \).

It is shown in Appendix B that for small \( \delta \omega \)

\[
K_0 = k_{0,2} \cdot \lambda^2 + k_{0,4} \cdot \lambda^4 + O(\lambda^6)
\]

(31)

\[
K_C = k_{C,2} \cdot \lambda^2 + k_{C,4} \cdot \lambda^4 + O(\lambda^6)
\]

(32)

\[
K_S = k_{S,3} \cdot \lambda^3 + O(\lambda^5)
\]

(33)

where

\[
\lambda = N \cdot \delta \omega
\]

(34)

and the coefficients \( k_{0,2}, k_{0,4}, k_{C,2}, k_{C,4}, \) and \( k_{S,3} \) are given in the Appendix.

Note that the dependence of the CR frequency bound \( B_0 \) on \( \delta \omega \) is given through the product \( (N \cdot \delta \omega) \) for small \( \delta \omega \).

Use of (31)–(34) in (17), (20), and (21) gives, respectively

\[
\phi = p \cdot (N \cdot \delta \omega) + \Omega((N \cdot \delta \omega)^3)
\]

(35)

\[
(B_0)_{\text{max}} = \frac{M}{\text{SNR} \cdot N^3} \cdot (N \cdot \delta \omega)^4 + O((N \cdot \delta \omega)^2)
\]

(36)

\[
(B_0)_{\text{min}} = \frac{m}{\text{SNR} \cdot N^3} \cdot (N \cdot \delta \omega)^2 + O(1)
\]

(37)
for small $\delta\omega$, where

$$p = \frac{k_{\delta\omega}}{2k_{0,2}}$$  \hspace{1cm} (38)

$$M = \frac{1}{k_{0,4} - k_{\delta\omega} - \frac{k_{\delta\omega}^2}{2k_{0,2}}}$$  \hspace{1cm} (39)

$$m = \frac{1}{2k_{0,2}}.$$  \hspace{1cm} (40)

The results (36) and (37) show that the largest CR bound $B_k$ on $\text{var} \{\tilde{\omega}_0\}$ is proportional to $(N \cdot \delta\omega)^{-1}/N^3$, whereas the smallest CR bound $B_0$ on $\text{var} \{\tilde{\omega}_0\}$ is proportional to $(N \cdot \delta\omega)^{-2}/N^3$ for small frequency separations $\delta\omega$.

Note that the inverse power dependence of the asymptotic bounds on $\delta\omega$ is through $(N \cdot \delta\omega)$, indicating that the bound will be large, provided that $(N \cdot \delta\omega)$, rather than $\delta\omega$, is small. This shows the importance of a sufficient number of samples when $\delta\omega$ is small.

The results in (36) and (37) are consistent with the empirical result of Swingler, which states that when the bounds are averaged over the phase, the resultant “pseudo-bound” is proportional to $(n \cdot \delta\omega)^{-3}$ for small $\delta\omega$ [9].

Examples show that the results (35)–(37) closely approximate the exact ones whenever the frequency separation is smaller than the Fourier resolution limit

$$\delta\omega < \frac{2\pi}{N}$$  \hspace{1cm} (41)

which is the range considered with the high-resolution frequency estimators.

Example 1: Consider the common choice of the first sample index $n = 1$. Straightforward (albeit tedious) calculation shows that

$$p = \frac{(N+1)}{4N}$$  \hspace{1cm} (42)

$$M = 201000 \cdot G_N$$  \hspace{1cm} (43)

$$m = 300 \cdot g_N$$  \hspace{1cm} (44)

where

$$G_N = \frac{N^6}{(N^2 - 9)(N^2 - 4)(N^2 - 1)}$$  \hspace{1cm} (45)

$$g_N = \frac{N^4}{(N^2 - 4)(N^2 - 1)}.$$  \hspace{1cm} (46)

Note that the penalty terms in (45) and (46) are consistent with the necessary conditions given in (22) on the number of samples $N$ for the invertibility of the Fisher information matrix. In addition, note that $G_N$ and $g_N$ approach one as $N \to \infty$.

Use of (42)–(44) in (35)–(37) gives

$$[(\phi_0)_{\text{max}}]_{\text{first term}} = -\frac{(N+1)}{4} \cdot \delta\omega + \frac{\pi}{2}$$  \hspace{1cm} (47)

$$[(\phi_0)_{\text{min}}]_{\text{first term}} = -\frac{(N+1)}{4} \cdot \delta\omega$$  \hspace{1cm} (48)

$$[(B_k)_{\text{max}}]_{\text{first term}} = \frac{201000 \cdot G_N \cdot (N \cdot \delta\omega)^{-4}}{\text{SNR} \cdot N^3}$$  \hspace{1cm} (49)

$$[(B_k)_{\text{min}}]_{\text{first term}} = \frac{300 \cdot g_N \cdot (N \cdot \delta\omega)^{-2}}{\text{SNR} \cdot N^3}.$$  \hspace{1cm} (50)

Fig. 1 shows the largest and smallest exact CR frequency bounds (20) and (21) for $N = 10$ samples. The vertical coordinate depicts the value of the product $B_k \cdot \text{SNR} \cdot N^3$; the horizontal coordinate depicts the value of $(\delta\omega/\Omega)$, where $\Omega$ denotes the Fourier resolution limit. The largest and smallest asymptotic bounds (49) and (50) are also shown in the figure. Note that the actual bounds closely follow the asymptotic bounds in the interval $(\delta\omega/\Omega) < 1$. In addition, note that the difference between the two limits of the bound is large in the region $(\delta\omega/\Omega) \ll 1$, indicating the strong dependence of the bound on the phase in this region. For $(\delta\omega/\Omega) > 1$, the difference becomes small, and the dependence of the bound on the phase may be neglected.

Fig. 2 shows the exact worst and best case values of the phase given by (15) and (16) versus the normalized frequency difference, $\delta\omega/\Omega$, for the interval $(\delta\omega/\Omega) < 1$ of interest. Their first terms given by (47) and (48) are also shown in the figure. Note that the first terms are sufficient to accurately determine the actual critical values of the phase.
The worst-case value of the phase $(\phi_0)_{\text{max}} \to \pi/2$ and the best-case value of the phase $(\phi_0)_{\text{min}} \to 0$ as $\delta \omega \to 0$. These limit values of the critical phases are consistent with the observations in [9] on worst- and best-case estimation scenarios for the real model with small frequency separations. The worst-case scenario occurs when the samples are “concentrated” near a zero crossing of the sinusoid (where it is most linear), and the best-case scenario occurs when they are near a peak of the sinusoid (where it is most nonlinear).

The asymptotic bounds (49) and (50) may be used to estimate the two limits of the actual CR frequency bound for $(\delta \omega/\Omega) < 1$. The relative errors in these estimates are shown in Figs. 3 and 4 for the cases of numbers of samples between 10 and 1000 and normalized frequency separations between 0.1 and 1 that appear to be of practical interest; the relative error in an estimate is defined by $\text{rel. error} = (\text{exact} - \text{estimate})/\text{exact}$. Note that the relative errors are independent of the SNR. From the figures, we see that the errors are less than 5% for all the cases considered.

Note that the simple expressions (49) and (50) corresponding, respectively, to the largest and smallest asymptotic bounds are independent of the first sample index $n_\ast$.

V. FREQUENCY RESOLUTION

For frequency separations less than the Fourier resolution limit, the results in (36) and (37) can be used to determine the minimum (threshold) SNR at which an unbiased estimator can resolve the two frequencies. If an unbiased estimator attaining the bound exists, it will resolve the frequencies with high probability when the root CR bound is smaller than half the frequency separation

$$\sqrt{E[1]} \leq \frac{\delta \omega}{2}. \quad (51)$$

Squaring of (51) and substitutions of (36) and (37) give the threshold SNR $(\text{SNR}_T)$ in the range

$$4 \ln \left( \frac{N \cdot \delta \omega}{\Omega} \right) \leq \text{SNR}_T \leq 4M \cdot \left( \frac{N \cdot \delta \omega}{\Omega} \right)^{-6}. \quad (52)$$

Equation (52) shows that for small frequency separations, the threshold SNR necessary to resolve the frequencies is proportional to $(N \cdot \delta \omega)^{-6}/N$ for the worst value of the phase and is proportional to $(N \cdot \delta \omega)^{-4}/N$ for the best value of the phase.

The fact that the inverse power dependence of the SNR on $\delta \omega$ happens through $(N \cdot \delta \omega)$ for small $\delta \omega$ leads to rather low threshold values and supports the conclusion reached in [7] on the practicability of the high-resolution frequency estimators for the single experiment data case.

Example 2: For $N = 10$ and $\delta \omega = \frac{1}{3} \cdot (2\pi/N)$, (52) gives the threshold SNR’s between 2–20 dB as $\phi_0$ varies. Reducing the frequency separation further to $\delta \omega = \frac{1}{3} \cdot (2\pi/N)$ requires SNR’s between 9–30 dB.

For the above two cases, the mean square errors of a brute-force least-squares frequency estimator obtained from the minimum of the function

$$S(\phi_0, \phi_0, \omega_0) = \sum_{t=1}^{N} [y(t) - \alpha_0 \cos (\omega_0 t + \phi_0)]^2 \quad (53)$$

on a properly selected $(33 \times 33 \times 33)$-point grid in the $(\alpha_0, \phi_0, \omega_0)$-space are compared with the CR lower bound for worst and best estimation cases and for a range of SNR’s in Figs. 5 and 6, respectively. The mean square errors (which are shown by the discrete points on the figures) were estimated from 100 Monte Carlo runs performed for the estimator for a given estimation case and SNR. The simulation results of the figures indicate the possibility of high-resolution frequency estimation for practical values of the SNR.

VI. COMPLEX DATA CASE

The complex counterpart of the previous time-series data consist of two complex sinusoids and complex zero-mean
white Gaussian noise of variance $\sigma^2$

$$y(t) = \sum_{i=1}^{2} \alpha_i \exp[j(\omega_i t + \varphi_i)] + c(t)$$

$$t = n \cdots n + N - 1$$

(54)

where the real and positive amplitudes $\{\alpha_i\}$, the initial phase angles $\{\varphi_i\}$, the normalized frequencies $\{\omega_i\}$ of the sinusoids, and the variance $\sigma^2$ of the noise are unknown parameters to be estimated.

The Fisher information matrix $J$ depends on the frequencies and initial phase angles of the two sinusoids through the frequency difference $\delta \omega = (\omega_1 - \omega_2)$ and the corresponding initial phase difference $\delta \varphi = (\varphi_1 - \varphi_2)$.

We prove the following theorem.

**Theorem 2:** Let the data be given by (54). Then, the CR bounds $B_i$ on $\text{var} \{\hat{\omega}_i\}, i = 1, 2$ are given by

$$B_i = \frac{1}{\text{SNR}_i \cdot N^3} \frac{X_0 + X_C \cdot \cos(2 \cdot \delta \varphi) + X_S \cdot \sin(2 \cdot \delta \varphi)}{K_0}.$$  

(55)

$\text{SNR}_i$ denotes the SNR for the $i$th sinusoid

$$\text{SNR}_i = \frac{\alpha_i^2}{\sigma^2}$$

(56)

$N$ denotes the number of data samples, and

$$X_0 = 2K_0 - \frac{K_0^2 + K_S^2}{K_0}$$

(57)

$$X_C = \frac{K_S^2 - K_S^2}{K_0}$$

(58)

$$X_S = -\frac{2K_C K_S}{K_0}$$

(59)

where the $K_0, K_C$, and $K_S$ are given by (7)–(9) with $\delta \omega$ replaced by $(\omega_1 - \omega_2)$.

**Proof:** The proof is given in Appendix C.

The result in (55) gives the CR frequency bound $B_i$ as a simple function of the phase difference $\delta \varphi$ between the two sinusoids. We observe from (55) that the bound is periodic in $\delta \varphi$ with a period of $\pi$, and we consider $\delta \varphi$ in the interval

$$I' = \left\{ \delta \varphi : \delta \varphi \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \right\}.$$  

(60)

We have the following corollary:

**Corollary 2:** The bound $B_i$ that is considered to be a function of the phase difference $\delta \varphi$ has one maximum point and one minimum point in the interval $I'$ given by, respectively

$$\left( \delta \varphi \right)_{\text{max}} = \left\{ \delta \varphi : \delta \varphi = \frac{\pi}{2} \cdot \frac{X_S}{X_C} \right\} \text{ if } X_C \geq 0$$

$$\left( \delta \varphi \right)_{\text{min}} = \left\{ \delta \varphi : \delta \varphi = \frac{\pi}{2} \cdot \frac{X_S}{X_C} \right\} \text{ if } X_C < 0$$

(61)

and

$$\left( \delta \varphi \right)_{\text{max}} = \left\{ \delta \varphi : \delta \varphi = \frac{\pi}{2} \cdot \frac{X_S}{X_C} \right\} \text{ if } X_C \geq 0$$

$$\left( \delta \varphi \right)_{\text{min}} = \left\{ \delta \varphi : \delta \varphi = \frac{\pi}{2} \cdot \frac{X_S}{X_C} \right\} \text{ if } X_C < 0$$

(62)

where

$$\phi' = \frac{1}{2} \arctan \left( \frac{X_S}{X_C} \right)$$

(63)

$$\arctan(\cdot) \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$$

(64)

$$\text{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases}$$

(65)

Furthermore, the maximum and the minimum values of $B_i$ are given by, respectively

$$\left( B_i \right)_{\text{max}} = \frac{1}{\text{SNR}_i \cdot N^3} \frac{X_0 + X_C \cdot \cos(2 \cdot \delta \varphi) + X_S \cdot \sin(2 \cdot \delta \varphi)}{K_0}$$

$$= \frac{1}{\text{SNR}_i \cdot N^3 \cdot 2 \cdot (K_0^2 - K_C^2 - K_S^2)}$$

(66)
and

\[(B_0)_{\text{min}} = \frac{1}{\text{SNR}_i} \cdot \frac{N^3}{X_0 + \sqrt{X_0^2 + X_S^2}} \cdot \frac{1}{\text{SNR}_i \cdot N^3} \cdot \frac{2}{K_0}. \quad (67)\]

The values of the phase difference \(\delta \phi\) given by (61) and (62) give, respectively, the largest and smallest CR bound on the variance of any unbiased estimator \(\hat{\phi}^i\) of the frequency \(\omega^i\), and, in addition, as the expression for the inverse Fisher information matrix \(J^{-1}\) derived in Appendix C shows, the largest and smallest values of the \(d_{ij}(J^{-1})\).

Equations (57)–(59) give the relationship between the CR frequency bounds for the real and complex data cases. It is clear from this relationship that the necessary conditions (22) for the invertibility of the Fisher information matrix in the real case carry over the complex case as

\[N > 2, \quad \text{and} \quad N = 3 \Rightarrow \delta \phi \neq (\delta \phi)_{\text{max}}. \quad (68)\]

For the symmetric sampling case of \(N\) is odd and \(\eta = -(N - 1)/2\), it follows from (26) and (57)–(59) that

\[X_0 = 2K_0 - \frac{K_C^2}{K_0} \quad (69)\]

\[X_C = -\frac{K_C^2}{K_0} \quad (70)\]

\[X_S = 0. \quad (71)\]

In this case, note from (70) that \(X_C < 0\) so that the worst and best phase difference expressions become independent of the frequency separation and the number of samples

\[(\delta \phi)_{\text{max}} = 0 \quad (72)\]

\[(\delta \phi)_{\text{min}} = \frac{\pi}{2} \quad (73)\]

Clearly, (72) and (73) represent, respectively, simple worst-and best-case scenarios in testing the performance of an algorithm.

For small \(\delta \omega\), use of (31)–(34) with \(\delta \omega = (\omega_1 - \omega_2)\) in (63), (66), and (67) gives, respectively

\[\phi' = p' \cdot (N \cdot \delta \omega) + O\{(N \cdot \delta \omega)^2\} \quad (74)\]

\[(B_i)_{\text{max}} = \frac{M'}{\text{SNR}_i \cdot N^3} \cdot (N \cdot \delta \omega)^{-1} + O\{(N \cdot \delta \omega)^{-2}\} \quad (75)\]

\[(B_i)_{\text{min}} = \frac{m'}{\text{SNR}_i \cdot N^3} \cdot (N \cdot \delta \omega)^{-2} + O\{1\} \quad (76)\]

where

\[p' = 2p \quad (77)\]

\[M' = \frac{M}{4} \quad (78)\]

\[m' = m. \quad (79)\]

Equations (78) and (79) show that for the same SNR \(N\) and \(\delta \omega\), the largest CR frequency bound in the complex case is four times smaller than the largest bound in the real case for small \(\delta \omega\), whereas the smallest bounds in the two cases are the same. Thus, the worst phase threshold SNR necessary to resolve the two frequencies, when their separation is small, is 6 dB lower in the complex case, whereas the best phase threshold SNR’s of the two cases are the same.

The smaller variation of the bound in the complex case has also been reported in [9] based on numerical comparisons.

**Example 3:** For the common choice of the first sample index \(n = 1\), we have

\[\left[(\delta \phi)_{\text{max}}\right]_{\text{first term}} = \left(\frac{N + 1}{2}\right) \cdot \frac{\delta \omega}{\sqrt{\text{SNR}_i \cdot N^3}} \quad (80)\]

\[\left[(\delta \phi)_{\text{min}}\right]_{\text{first term}} = \left(\frac{N + 1}{2}\right) \cdot \frac{\delta \omega}{\sqrt{\text{SNR}_i \cdot N^3}} + \frac{\pi}{2} \quad (81)\]

\[\left[(B_i)_{\text{max}}\right]_{\text{first term}} = \frac{50 \cdot 400 \cdot G_N \cdot (N \cdot \delta \omega)^{-1}}{\text{SNR}_i \cdot N^3} \quad (82)\]

\[\left[(B_i)_{\text{min}}\right]_{\text{first term}} = \frac{300 \cdot G_N \cdot (N \cdot \delta \omega)^{-2}}{\text{SNR}_i \cdot N^3}. \quad (83)\]

The largest and the smallest exact CR frequency bounds (66) and (67) versus the normalized frequency separation \(\delta \omega/\Omega\) are shown in Fig. 7, together with the corresponding asymptotic bounds (82) and (83) for \(N = 10\) samples. We see that the actual bounds closely follow the asymptotic bounds in the interval \((\delta \omega/\Omega) < 1\). Note also that the bound strongly depends on the phase difference for \((\delta \omega/\Omega) < 1\). The smaller variation of the bound in the complex case becomes clear from the comparison of the figure with the corresponding one (Fig. 1) for the real case.

The exact worst- and best-case values of the phase difference given by (61) and (62) and their one-term approximations given by (80) and (81) are shown in Fig. 8. We see that the one-term approximations adequately determine the actual critical phase differences.

Note that the worst-case and the best-case limit values of the phase difference are reversed in the complex case: \((\delta \phi)_{\text{max}} \rightarrow 0\) and \((\delta \phi)_{\text{min}} \rightarrow \pi/2\) as \(\delta \omega \rightarrow 0\).

Simulations by Wilkes and Cadzow for this case (the complex model with \(n = 1\)) for a variety of high-resolution frequency estimators indicated that the variance of the estimators was significantly larger at the worst-case value (80) of the phase difference [10].
Figs. 9 and 10 show the relative errors in the asymptotic bounds (82) and (83) as approximations, respectively, to the largest and smallest exact CR bounds for the cases of $N$ between 10 and 1000, and normalized frequency separations between 0.1 and 1. As we see from the figures, the asymptotic bounds closely approximate the actual bounds in all the cases considered.

A. Rearranging the Complex Model and Relation to Previous Results

The nonmatrix expression (55) derived herein for the complex model CR bound can also be obtained from the Stoica and Nehorai formula [5, (4.1)]. To do so, one needs to recast the data model defined by (54) in the form of the more general data model of [5] in such a way that their formula applies.

To this end, let us define the vectors

\[ y ≜ [y(n) \cdots y(n + N - 1)]^T \]  
\[ a(ω) ≜ [e^{jωn} \cdots e^{j(n+N-1)ω}]^T \]  
\[ x ≜ [\alpha_1 e^{j\alpha_1} \cdots \alpha_2 e^{j\alpha_2}]^T \]  
\[ c ≜ [c(n) \cdots c(n + N - 1)]^T \]

where $(\cdot)^T$ denotes the transpose. Then, (54) can be rewritten as

\[ y = [a(ω_1) a(ω_2)] \cdot x + c. \]  

The rearrangement of the complex model in (88) is in the form of the data model of [5], satisfies the necessary conditions A1, A2, and AML of [5], and is the one for which the derivation given in [5, Appendix E] holds true.

Their derivation, however, still differs from ours given in Appendix C in that it is based on a Fisher information matrix that is not directly applicable to the amplitudes and phases of the sinusoids.

The small $\delta ω$ results derived herein for the complex model can also be deduced to some extent (contrary to the belief in [7]) from the results of [6]. The results of [6] need the foregoing assumptions of [5] and the additional assumption that the matrix $\text{Re} [x^H]$ is invertible, where $x$ is given by (86), and $(\cdot)^H$ denotes the conjugate transpose. It is easy to show that this matrix is nonsingular for $\delta \varphi \neq 0$ (in the interval $\mathbb{F}$).

Thus, we assume that $\delta \varphi \neq 0$. The result in [6, (33)] then gives

\[ [B_i]_{\text{first term}} = \frac{1}{SNR_i} \frac{(N \cdot \delta ω)^2}{N^3 k_0 \omega [1 - \cos(2 \cdot \delta \varphi)]} \]  

which is, of course, identical to the first term of the result one would get from the substitution of (31)–(34) into (55).
Specifically, one gets
\[
(B_3 \cdot \text{SNR}_i \cdot N^3)^{-1} = k_{0,2} \left[ 1 - \cos(2 \cdot \delta \varphi) \right] \cdot (N \cdot \delta \omega)^2 \left[ 2k_{2,3} \sin(2 \cdot \delta \varphi) \right]
\]
\[
\cdot (N \cdot \delta \omega)^3 + \left( 3k_{0,4} - 2k_{2,4} - \frac{k_{2,3}^2}{k_{0,2}} \right) \cdot \cos(2 \cdot \delta \varphi) \cdot (N \cdot \delta \omega)^4
\]
\[
+ \left( k_{0,4} - 2k_{2,4} + \frac{k_{2,3}^2}{k_{0,2}} \right) \cdot \cos(2 \cdot \delta \varphi) \cdot (N \cdot \delta \omega)^4 + O((N \cdot \delta \omega)^5).
\]
(90)

Note that for $\delta \varphi = 0$, the first two terms of (90) are zero, and the third term becomes the dominant one. In addition, note that although the first term of (90) is sufficient to determine the smallest asymptotic bound $(B_3)_{\text{min}}$, determination of the largest asymptotic bound $(B_3)_{\text{max}}$ requires the first three terms of the series (to account for all the $O((N \cdot \delta \omega)^4)$ terms). Only the first term of the series, however, is (readily) available in [6].

Swingler, in his later work, where he considers the signal phases as deterministic parameters, as in here, has derived, in an entirely empirical way, approximate expressions for the bounds for the two models [11, (3) and (8)]. We note here that these approximate expressions have the same form as the corresponding exact expressions (55) and (5), respectively.

Finally, we remark that it follows from the decoupling property of the CR bounds (see, for example, [12, Section 13.4]) that the small $\delta \omega$ results derived herein for the two-signal cases are also applicable to the multiple-signal cases, provided that the other signals are “well-separated” from the closely spaced pair under consideration.

VII. CONCLUSION

This paper presented exact nonmatrix expressions for the CR frequency bounds applicable to the time-series data models consisting of either a single real sinusoid in real white Gaussian noise or two complex sinusoids in complex white Gaussian noise in the case of a single snapshot containing a finite number $N$ of data samples. The dependence of the bound on the initial phase difference between the two signal components is examined, and the largest and smallest bounds and the corresponding critical values of the phase difference are obtained for each model.

The exact frequency bound expressions are analyzed for small frequency separations $\delta \omega$. The results show that the largest bound is proportional to $(N \cdot \delta \omega)^{-1}/N^3$ and that the smallest bound is proportional to $(N \cdot \delta \omega)^{-2}/N^3$ for small $\delta \omega$.

Based on the small $\delta \omega$ results, we found that the threshold SNR at which an unbiased frequency estimator is able to resolve the two signal frequencies with high probability is at least proportional to $(N \cdot \delta \omega)^{-1}N$ for the worst-phase case and to $(N \cdot \delta \omega)^{-1}/N$ for the best-phase case for small $\delta \omega$.

The fact that the inverse power dependence of the bound on $\delta \omega$ is through the product $(N \cdot \delta \omega)$ for small $\delta \omega$ results in moderate threshold SNR values, indicating the practicability of the high-resolution frequency estimators for time-series spectrum analysis.

APPENDIX A

PROOF OF THEOREM 1

Under the white Gaussian noise assumption, the Fisher information matrix for the unknown parameters is given by (see, for example, [13, Appendix A])

\[
J(\sigma^2, \theta_0, \varphi_0, \omega_0) = \begin{bmatrix}
N & 0 \\
0 & \frac{1}{\sigma^2} \sum_{t=1}^{n+N-1} c_1(t)c_1(t)^T
\end{bmatrix}
\]

where
\[
c_1(t) = \begin{bmatrix}
-\cos(\omega_0t + \varphi_0) \\
\omega_0t \sin(\omega_0t + \varphi_0)
\end{bmatrix}.
\]

(A.1)

The submatrix
\[
J_1 = \frac{1}{\sigma^2} \sum_{t=1}^{n+N-1} c_1(t)c_1(t)^T
\]

(A.2)

is the Fisher information matrix corresponding to the signal parameters.

Consider now the following decomposition of $c_1(t)$:

\[
c_1(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \omega_0 & 0 \\
0 & 0 & \omega_0
\end{bmatrix} \cdot \begin{bmatrix}
\cos \varphi_0 & \sin \varphi_0 & 0 \\
-\sin \varphi_0 & \cos \varphi_0 & 0 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
-\cos \omega_0t \\
\sin \omega_0t \\
t \sin(\omega_0t + \varphi_0)
\end{bmatrix}
\]

(A.3)

\[
\equiv D_1 \cdot Q_1 \cdot \varepsilon_1(t).
\]

(A.4)

It follows from (A.4) that
\[
\sum_{t=1}^{n+N-1} c_1(t)c_1(t)^T = D_1 \cdot Q_1 \cdot \begin{bmatrix}
\sum_{t=1}^{n+N-1} \varepsilon_1(t)\varepsilon_1(t)^T
\end{bmatrix} \cdot Q_1^T \cdot D_1.
\]

(A.5)

Straightforward calculation gives
\[
\sum_{t=1}^{n+N-1} \varepsilon_1(t)\varepsilon_1(t)^T = \frac{1}{2} \cdot K_1 \cdot J_1 \cdot K_1
\]

(A.6)

where
\[
K_1 = \begin{bmatrix}
\sqrt{N} & 0 & 0 \\
0 & \sqrt{N} & 0 \\
0 & 0 & N\sqrt{N}
\end{bmatrix}
\]

(A.7)

\[
I_1 = \begin{bmatrix}
A_1 \\
B_1 \\
C_1
\end{bmatrix}
\]

(A.8)

\[
A_1 = \begin{bmatrix}
\Gamma_0 + C_0 & -S_0 \\
-S_0 & \Gamma_0 - C_0
\end{bmatrix}
\]

(A.9)

\[
b_1 = \frac{-S_1 \cos \varphi_0 - (\Gamma_1 + C_1) \sin \varphi_0}{(\Gamma_1 - C_1) \cos \varphi_0 + S_1 \sin \varphi_0}
\]

(A.10)

\[
c_1 = \Gamma_2 - C_2 \cos(2\varphi_0) + S_2 \sin(2\varphi_0)
\]

(A.11)
It follows from (A.3), (A.5), and (A.6) that

\[ J_1 = \frac{1}{2\sigma^2} \cdot D_1 \cdot Q_1 \cdot K_1 \cdot I_1 \cdot K_1 \cdot Q_1^T \cdot D_1 \]  
(A.12)

so that

\[ J_1^{-1} = 2\sigma^2 \cdot D_1^{-1} \cdot Q_1 \cdot K_1^{-1} \cdot I_1^{-1} \cdot K_1^{-1} \cdot Q_1^T \cdot D_1^{-1} \]  
(A.13)

using the property that \( Q_1 \) is orthogonal.

It follows from (A.1), (A.13), and application of the matrix inversion lemma to \( I_1 \) in (A.8) that

\[ B_0 = \frac{2\sigma^2}{\alpha_0^2} \cdot \frac{1}{N^3} \cdot (c_1 - b_1^T A_1^{-1} b_1)^{-1}. \]  
(A.14)

From (A.9)–(A.11), we find

\[ c_1 - b_1^T A_1^{-1} b_1 = K_0 + K_C \cdot \cos (2\varphi_0) + K_S \cdot \sin (2\varphi_0), \]  
(A.15)

Substitution of (A.15) into (A.14) gives the result (5).

### Appendix B

**Derivation of (31)–(34)**

The functions \( C_r, S_r, r = 0, 1, 2 \) can be expressed as follows in terms of Taylor series about \( \delta \omega = 0 \):

\[ C_r = \sum_{k=0}^{\infty} \left( \frac{(-1)^k}{(2k)!} \right) \lambda^{2k} \]  
(B.1)

\[ S_r = \sum_{k=0}^{\infty} \left( \frac{(-1)^k}{(2k+1)!} \right) \lambda^{2k+1} \]  
(B.2)

where for \( k \geq 0 \)

\[ \Gamma_k = \frac{1}{N^{k+1}} \sum_{\tau=-n}^{n} \tau^k \]  
(B.3)

and

\[ \lambda = N \cdot \delta \omega. \]  
(B.4)

Use of (B.1) and (B.2) in (7)–(9) gives

\[ K_0 = k_{0,2} \cdot \lambda^2 + k_{0,4} \cdot \lambda^4 + O(\lambda^6) \]  
(B.5)

\[ K_C = k_{C,2} \cdot \lambda^2 + k_{C,4} \cdot \lambda^4 + O(\lambda^6) \]  
(B.6)

\[ K_S = k_{S,3} \cdot \lambda^3 + O(\lambda^5) \]  
(B.7)

for small \( \lambda \), where

\[ k_{0,2} = \frac{(\Gamma_0^2 + 4 \cdot \Gamma_0 \cdot \Gamma_2 - 4 \cdot \Gamma_2^2)}{4 \cdot (\Gamma_0^2 - \Gamma_2^2)} \]  
(B.8)

\[ k_{0,4} = (-2 \cdot \Gamma_0^2 \cdot \Gamma_4 + 4 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 - \Gamma_2^2 \cdot \Gamma_4 + 4 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 + \Gamma_0^2 \cdot \Gamma_4 - 6 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 + 4 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 + 12 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 - 12 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 - 2 \cdot \Gamma_0^2 \cdot \Gamma_4 + 10 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 + 4 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 + 1 \cdot \Gamma_2^2 + 30 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 - 9 \cdot \Gamma_0^2) / 144 \cdot (\Gamma_0^2 - \Gamma_2^2) \]  
(B.9)

\[ k_{C,2} = k_{0,2} \]  
(B.10)

\[ k_{C,4} = (-4 \cdot \Gamma_0^2 \cdot \Gamma_4 + 2 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 + \Gamma_0^2 \cdot \Gamma_4 + 2 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 + 12 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 - 12 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 - 2 \cdot \Gamma_0^2 \cdot \Gamma_4 + 10 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 + 4 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 + 1 \cdot \Gamma_2^2 + 30 \cdot \Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 - 9 \cdot \Gamma_0^2) / 144 \cdot (\Gamma_0^2 - \Gamma_2^2) \]  
(B.9)

and

\[ k_{S,3} = \frac{12 \cdot (\Gamma_0 \cdot \Gamma_2 - \Gamma_2^2)}{(\Gamma_0 \cdot \Gamma_2 + \Gamma_2^2)} \]  
(B.12)

### Appendix C

**Proof of Theorem 2**

Under the white Gaussian noise hypothesis, the Fisher information matrix for the unknown parameters is given by (see, for example, [13, Appendix A])

\[
J(\sigma^2, \omega_1, \varphi_1, \omega_2, \varphi_2, \omega_1, \omega_2) = \begin{bmatrix}
\frac{N}{\sigma^4} & 0 \\
0 & \frac{2}{\sigma^2} \sum_{t=-n}^{n-1} [\tilde{e}_2(t)\tilde{e}_2(t)^T + \tilde{e}_2(t)\tilde{e}_2(t)^T]
\end{bmatrix}
\]  
(C.1)

where

\[
e_2(t) = \begin{bmatrix}
-\exp[j(\omega t + \varphi_1)] \\
-j\alpha_1 \exp[j(\omega t + \varphi_1)] \\
-\exp[j(\omega t + \varphi_2)] \\
-j\alpha_2 \exp[j(\omega t + \varphi_2)] \\
-j\alpha_1 \exp[j(\omega t + \varphi_1)] \\
-j\alpha_2 \exp[j(\omega t + \varphi_2)]
\end{bmatrix}
\]  
(C.2)

and \( \tilde{e}_2(t) = \text{Re}[e_2(t)], \tilde{e}_2(t) = \text{Im}[e_2(t)] \).

Now, concentrate on the Fisher information matrix for the signal parameters

\[ J_2 = \frac{2}{\sigma^2} \sum_{t=1}^{n-N+1} [\tilde{e}_2(t)\tilde{e}_2(t)^T + \tilde{e}_2(t)\tilde{e}_2(t)^T], \]  
(C.3)

The vectors \( \tilde{e}_2(t) \) and \( \tilde{e}_2(t) \) can be decomposed as
It follows from (C.4) and (C.5) that

\[ \tilde{\varphi}_2(t) = \begin{bmatrix} 1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos \varphi_1 & \sin \varphi_1 & \cos \varphi_2 & \sin \varphi_2 \\ -\sin \varphi_1 & \cos \varphi_1 & -\sin \varphi_2 & \cos \varphi_2 \end{bmatrix} \]

\[ \equiv D_2 \cdot \varphi_2(t). \]

Straightforward (albeit tedious) calculation gives

\[ \sigma_2^2 \sigma_2^{-1} b_2 = K_0 \begin{bmatrix} \frac{K_0}{K_C \cos(\delta \varphi) - K_S \sin(\delta \varphi)} \\ -K_C \cos(\delta \varphi) - K_S \sin(\delta \varphi) \end{bmatrix} \]

\[ \equiv \begin{bmatrix} \frac{\sqrt{N}}{N \sqrt{N}} \\ \frac{\sqrt{N}}{N \sqrt{N}} \end{bmatrix} \]

\[ \begin{bmatrix} K_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_2 & b_2 \\ \Gamma_0 & 0 & C_0 & S_0 \\ 0 & \Gamma_0 & -S_0 & C_0 \\ S_0 & C_0 & 0 & \Gamma_0 \end{bmatrix} \]

\[ a_2 = \begin{bmatrix} \frac{1}{\sigma_2^2 \sigma_2^{-1}} \cdot \frac{1}{N^3} \cdot (\sigma_2^2 - \sigma_2^{-1} b_2)^{-1} \end{bmatrix} \]

References


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